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ON THE SELF-SIMILAR SOLUTION OF NAVIER-STOKES EQUATIONS WITH VOLUME SOURCES AND SINKS OF MASS

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Unlike the investigations in [1, 2] of the motion of fluid with surface sources and sinks of mass (injection and suction), the flow is considered here in the presence of uniformly distributed mobile volume sources and sinks in flat and round channels. It is shown that far away from the inlet a self-similar solution of the system of equations of motion can be obtained. The results are applicable, for instance, to two-phase (vapor-liquid) streams with condensation or evaporation for small volume concentrations of the discrete phase and absence of phase slip.

1. The steady axisymmetric flow of fluid in pipes with volume sources or sinks of mass which move at the medium velocity, is defined by the system of equations

$$u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r^\alpha}\left[\frac{\partial}{\partial x}\left(r^\alpha \frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u_x}{\partial r}\right)\right] \qquad (1.1)$$

$$u_x \frac{\partial u_r}{\partial x} + u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu}{r^\alpha}\left[\frac{\partial}{\partial x}\left(r^\alpha \frac{\partial u_r}{\partial x}\right) + \frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u_r}{\partial r}\right) - \left(\frac{u_r}{r}\right)^\alpha\right]$$

$$\frac{\partial}{\partial x}(r^\alpha u_x) + \frac{\partial}{\partial r}(r^\alpha u_r) = -r^\alpha \frac{\varkappa}{\rho}$$

where u_x and u_r are velocity vector components in the longitudinal and radial directions, $\varkappa$ is the capacity of volume sources or sinks ($\varkappa > 0$ related to sinks, $\varkappa < 0$ to sources), $\alpha = 0$ for a flat channel, and $\alpha = 1$ for a round pipe.

Let us consider the case of $\varkappa = \text{const}$. We shall seek a self-similar solution for system (1.1) far from the tube inlet in a form that satisfies the equation of continuity

$$u_x = U\left(1 - \frac{N}{2} X\right) f'(\lambda), \quad u_r = \frac{R\varkappa (f(\lambda) - \lambda)}{2\rho\lambda^{\alpha/2}} \tag{1.2}$$

$$X = 2\,\nu x / UR^2, \qquad \lambda = 2^{\alpha-1} R^{1+\alpha}, \; N = \varkappa R^2/\rho\nu$$

where R is the radius of the round pipe or width of the flat cannel, U is the mean mass velocity at inlet, and N is a parameter which defines the intensity of volume sources or sinks.

Substituting (1.2) into the second of Eqs. (1.1), we find that $\partial p/\partial r$ is independent of x. Then the first of Eqs. (1.1) reduces to the ordinary differential equation

$$(\lambda^\alpha f'')' + \frac{N}{4}\left[f'^2 - (f - \lambda) f''\right] = \frac{k}{2}, \quad k = \frac{1}{1 - NX/2} \frac{\partial (p/\rho U^2)}{\partial X} \tag{1.3}$$

Boundary conditions for the considered problem are of the form

$$\lim_{\lambda\to 0} f\lambda^{-\alpha/2} = 0, \; \lim_{\lambda\to 0} f''\lambda^{\alpha/2} = 0, \; f(1) = 1, \; f'(1) = 0 \tag{1.4}$$

The first and third conditions imply that the radial velocity component at the axis and the channel wall vanishes, while the second and fourth conditions imply, respectively, that the axial velocity component is symmetric and vanishes at the wall.

Let us consider the solution of the boundary value problem (1.3), (1.4) when parameter N is very small or very great.

2. When $|N| \ll 1$ the form of function $f(\lambda)$ can be determined by the method of perturbations. Expanding f and k in series in powers of ε,

$$f = f_0 + \varepsilon f_1 + O(\varepsilon^2), \quad \frac{k}{2} = k_0 + \varepsilon k_1 + O(\varepsilon^2) \quad \left(\varepsilon = \frac{N}{4}\right) \tag{2.1}$$

and substituting formulas (2.1) into (1.3), (1.4), we obtain a system of equations for f_0 and f_1 with boundary conditions

$$(\lambda^\alpha f_0'')^1 = k_0, \quad (\lambda^\alpha f_1'')' = k_1 - f_0'^2 - \lambda f_0'' + f_0 f_0'' \tag{2.2}$$

$$\lim_{\lambda\to 0} f_i \lambda^{-\alpha/2} = 0, \quad \lim_{\lambda\to 0} f_i''\lambda^{\alpha/2} = 0, \; f_i(1) = 1 - i, \; f_i'(1) = 0, \; i = 0,1$$

The solution of problem (2.2) yields

$$f = \frac{3}{2}\lambda - \frac{\lambda^3}{2} + \frac{N}{4}\left(\frac{3}{70}\lambda - \frac{5}{56}\lambda^3 + \frac{\lambda^5}{20} - \frac{\lambda^7}{280}\right), \quad \alpha = 0 \tag{2.3}$$

$$f = 2\lambda - \lambda^2 + \frac{N}{4}\left(\frac{7}{18}\lambda - \frac{5}{6}\lambda^2 + \frac{\lambda^3}{2} - \frac{\lambda^4}{18}\right), \quad \alpha = 1$$

The presence of volume sources and sinks of mass results in the deviation of the flow from the Poiseuille flow which is defined by the two first terms un Eqs. (2.3). In comparison with the latter, its axial velocity profile is more prolate in the case of low intensity sinks $(N > 0)$, and in the case of sources $(N < 0)$ it is more filled. The dimensionless pressure gradient k is defined by formulas

$$k = -3 + {}^3/_7 N \quad \text{for } \alpha = 0, \; k = -4 + {}^7/_6 N \quad \text{for } \alpha = 1 \tag{2.4}$$

It follows from these that when $N > 0$ the rate of pressure drop is lower and for $N < 0$ it is more rapid. Thus the volume sources of mass increase and sinks decrease the over-all hydraulic resistance of channels.

3. To find the solution of Eq. (1.3) when $N \gg 1$ (intensive sinks are considered) we use the method of joining asymptotic expansions. We seek the external expansion far from the channel walls in the form of series

$$f = f_0 + \varepsilon f_1 + o(\varepsilon), \quad \frac{k}{2} = \frac{C_0}{\varepsilon^2} + \frac{C_1}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right) \quad \left(\varepsilon = \frac{2}{\sqrt{N}}\right) \tag{3.1}$$

Substituting in (1.3) for f and k their expressions (3.1) and equating terms of like powers of ε, we obtain

$$f_0'^2 - (f_0 - \lambda) f_0'' = C_0, \quad 2f_0'^2 f_1' - f_1 f_0'' - (f_0 - \lambda) f_1'' = C_1 \tag{3.2}$$

System (3.2) with boundary conditions

$$f_i(0) = 0, \quad f_i'(0) = 0, \quad f_i(1) = 1 - i, \quad i = 0,1$$

satisfies formulas

$$f = \lambda + {}^1/_2\, \varepsilon C_1 \lambda, \quad C_0 = 1 \tag{3.3}$$

The constant C_1 must be determined by the matching with the inner expansion. To derive the latter we introduce new variables and set k equal to the first term of external expansion. As the result we can rewrite Eq. (1.3) with an accuracy to within ε in the form

$$\varphi''' + \xi\varphi' = \varphi\varphi'' - \varphi'^2 + 1 \tag{3.4}$$

$$\xi = \frac{1-\lambda}{\varepsilon}, \quad \varphi = \frac{1-f}{\varepsilon}, \quad k = \frac{2C_0}{\varepsilon^2} = \frac{2}{\varepsilon^2}$$

In this approximation the equations for the flat and round channels are the same. The boundary conditions in new variables are

$$\varphi = \varphi' = 0 \quad \text{for} \quad \xi = 0 \tag{3.5}$$

Considering the right-hand side of (3.4) as an inhomogeneity we transform the differential equation with conditions (3.5) to the integral equation

$$\varphi = -\int_0^\xi \Phi\, d\zeta - \xi \int_0^\xi d\zeta \Phi \exp\left(\frac{\zeta^2}{2}\right) \int_0^\zeta \exp\left(-\frac{\eta^2}{2}\right) d\eta + \tag{3.6}$$

$$\left[\int_0^\xi \Phi \exp\left(\frac{\zeta^2}{2}\right) d\zeta + B\right] \int_0^\xi d\zeta \int_0^\zeta \exp\left(-\frac{\eta^2}{2}\right) d\eta \quad (\Phi = \varphi\varphi'' - \varphi'^2 + 1)$$

by using the method of variation of arbitrary constants. Equation (3.6) can be solved by iterations. Constants B and C_1 are determined by the condition of joining asymptotic expansions far from and close to the wall

$$\varphi(\xi \to \infty) = \xi - {}^1/_2 C_1$$

To estimate constants we restrict ourselves to the zero approximation. The substitution of $\varphi^\circ = \xi$ into (3.6) yields

$$B = \sqrt{2/\pi}, \quad C_1 = 2\sqrt{2/\pi}$$

The dimensionless pressure gradient k, determined by formula (1.3), now becomes

$$k = N/2 + 2\sqrt{2N/\pi}$$

which shows that intensive sinks ($N \gg 1$) induce pressure increase along the channel.

The derived solution indicates that the flow pattern in pipes is similar to that of the flow in the boundary layer. The axial velocity profile at the core of the stream tends to be uniform, while close to the walls there is a thin boundary layer of constant thickness of order $1/\sqrt{N}$.

Note that the region of existence of self-similar solutions is semi-infinite and bounded by the cross section where the flow rate vanishes, which in the case of sources corresponds to $X > 2 / N$ and in that of sinks to $-X < 2 / N$.

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ON THE PROBLEM OF DESALINIZATION OF SOIL CONTAINING READILY SOLUBLE SALTS

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A general solution of the problem of desalinization of soil containing rapidly soluble salts is given. The salts are initially nonuniformly distributed, and it is assumed that they pass instantaneously from the solid phase to the solution. A condition of the third kind is postulated at the soil surface, reflecting the continuity of the mass flux of the salts.

A different approach was used in [1] to construct a solution for a particular case of homogeneous salinization, and the problem of uniqueness of the solution was studied. The process of diffusion of salts in the course of washing the soil was also studied in [2].

The mathematical formulation of the problem has the form:

$$\begin{gathered} Dc_{\xi\xi} - vc_{\xi} = mc_{\tau}, \quad 0 < \xi < \xi_0(\tau) \\ -Dc_{\xi} + vc = vc_n, \quad \xi = 0 \\ Dc_{\xi} = \varphi(\xi)\, d\xi_0(\tau) / d\tau, \quad \xi = \xi_0(\tau) \end{gathered} \tag{1}$$

Here the diffusion coefficient D, rate of filtration v, porosity m and concentration c_n of the wash water are all assumed constant; $\xi_0(\tau) = v\tau / m$ denotes the front of the flow of water, ξ is the coordinate counted from the soil surface, τ is time, $c(\xi, \tau)$ is the concentration of the solution in motion and $\varphi(\xi)$ is an arbitrarily prescribed function of the initial bulk salinity of the soil. The latter function is subjected to the usual constraints imposed on the original of a Laplace transform.

Introducing the dimensionless variables x, t and the functions $u(x, t)$, we can reduce the problem (1) to the form

$$u_{xx} = u_t, \quad 0 < x < t, \; u_x - u / 2 = 0, \quad x = 0 \tag{2}$$

$$u_x + u / 2 = f(t) \exp(-t / 4), \quad x = t \tag{3}$$

$$x = \frac{v\xi}{D}, \quad t = \frac{v^2\tau}{mD}, \quad f(t) = \frac{1}{m}\varphi\left(\frac{Dt}{v}\right)$$